Technical Note: A Physics-Like View of Hadamard Mixing Systems

1. Intuition: a system with a built-in symmetry

Imagine a system where every signal can be split into two parts that are supposed to stay completely separate. Think of it like two “channels” or two “states” that shouldn’t interact.

Now imagine sending that signal through some process:

- If the process respects the split, the two parts stay independent
- If it doesn’t, the parts start to mix

That gives you a very natural idea:

> Measure how much mixing happens to understand how the system behaves.

This is exactly what the Hadamard-based construction gives you.

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2. The Hadamard transform as a symmetry

The Hadamard transform creates a clean split of the space into two halves:

- one half behaves like a “positive” state
- the other like a “negative” state

You can think of this like a simple symmetry in physics, such as:

- even vs odd
- spin up vs spin down

A system respects this symmetry if it keeps the two halves separate.

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3. Systems that respect or break the symmetry

When a system preserves the split:

- each half evolves independently
- nothing crosses between them

When it doesn't:

- energy or information flows between the halves
- the symmetry is effectively broken

This gives you a very useful diagnostic:

> mixing between the halves tells you how much the system violates the symmetry.

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4. Splitting any system into "good" and "bad" parts

Any system can be separated into two pieces:

- one part that respects the symmetry
- one part that causes mixing

You can even isolate or remove the mixing part if you want to enforce the structure.

This is very similar to how physics separates:

- a clean, symmetric system
- and a perturbation or interaction that breaks that symmetry

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5. Adding randomness: diagonal sign flips

Now introduce a very simple operation:

- flip the sign of each coordinate independently (randomly +1 or −1)

At first glance, this looks trivial. But when combined with the Hadamard structure, it becomes powerful.

This operation can also be split into:

- a part that preserves the symmetry
- a part that mixes the two halves

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6. What happens in practice

For a typical random choice of sign flips:

- both parts are large
- roughly half the effect preserves the split
- the other half mixes between the halves

So the system is doing two things at once:

- shuffling things within each half
- and exchanging information between the halves

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7. Two types of mixing

It helps to picture two “rooms”:

- one room for each half of the space

Then:

- the symmetry-preserving part rearranges things inside each room
- the symmetry-breaking part moves things between rooms

A random sign pattern does both strongly.

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8. Dynamics: local randomness plus global mixing

Now consider a single step of the system:

- first apply random sign flips (local randomness)
- then apply the Hadamard transform (global mixing)

This combination is very important:

> local randomness gets spread across the entire system

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9. Iteration: a simple dynamical system

If you repeat this process multiple times:

- each step introduces new randomness
- each step redistributes everything globally

Over time:

- structure is lost
- information spreads evenly
- the system becomes well mixed

This is very similar to what happens in physical systems that “thermalize.”

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10. Connection to rotations

There’s a geometric way to think about this:

- both the Hadamard transform and sign flips can be viewed as reflections
- combining reflections produces rotations

So each step of the system behaves like:

> a collection of small rotations in different directions

Repeating the process builds up more complex motion, like repeatedly rotating in changing planes.

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11. A physics-style interpretation

You can think of the system in physics terms:

- the Hadamard transform defines a symmetry
- the two halves are like two states or sectors
- symmetry-preserving parts keep those states separate
- symmetry-breaking parts allow transitions between them

Random sign flips act like:

- a random local “field” or disturbance

The full system becomes:

a simple model of a system with symmetry, perturbations, and mixing dynamics

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12. What the “leakage” idea becomes

In this language:

- leakage measures how much the symmetry is broken
- zero leakage means perfect conservation
- nonzero leakage means interaction between the two halves

So you’ve effectively defined a measurable quantity that tells you how the system behaves.

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13. Why this works so well

Even though the pieces are simple:

- sign flips are local and cheap
- the Hadamard transform is fast

their combination:

- spreads information everywhere
- removes structure
- behaves similarly to a fully random system

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14. Big picture

You can think of the whole setup as a “toy physics system”:

- a state evolves over time
- there is a built-in symmetry
- there are interactions that break the symmetry
- repeated evolution produces mixing and randomness

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15. Final intuition

A simple way to picture it:

- the Hadamard transform defines two separate channels
- random sign flips both shuffle within channels and connect them
- repeated steps act like random rotations and mixing

So from very simple ingredients, you get:

> a system that behaves like a complex, highly mixed, almost random process.